ELSEVIER

Linear Algebra and its Applications 341 (2002) 129–142

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Commutativity of projectors

Jerzy K. Baksalary [a], Oskar Maria Baksalary [b,*]

[a] *Department of Mathematics, Tadeusz Kotarbiński Pedagogical University, pl. Słowiański 9, PL 65-069 Zielona Góra, Poland*

[b] *Institute of Physics, Faculty of Physics, Adam Mickiewicz University, ul. Umultowska 85, PL 61-614 Poznań, Poland*

Received 9 February 2001; accepted 31 March 2001

Submitted by C.-K. Li

Dedicated to Professor Tsuyoshi Ando on the occasion of his 70th birthday

Abstract

It is known that necessary and sufficient conditions for the sum $\mathbf{P}_1 + \mathbf{P}_2$ and the difference $\mathbf{P}_1 - \mathbf{P}_2$ of projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ to be also projectors are $\mathbf{P}_1\mathbf{P}_2 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$, respectively, independently of whether $\mathbf{P}_1$ and $\mathbf{P}_2$ are orthogonal or not. The situation changes when considering the products of $\mathbf{P}_1$ and $\mathbf{P}_2$: in case of orthogonal projectors the condition $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$ is both necessary and sufficient for $\mathbf{P}_1\mathbf{P}_2$ (and thus for $\mathbf{P}_2\mathbf{P}_1$) to be a projector, but in the general case it discontinues to be necessary even if $\mathbf{P}_1\mathbf{P}_2$ along with $\mathbf{P}_2\mathbf{P}_1$ are required to be projectors. The purpose of the present paper is to investigate similarities and dissimilarities of this kind between several results concerning orthogonal projectors and their counterparts corresponding to arbitrary projectors, with special emphasis laid on the commutativity condition. The investigations refer to matrix representations of projectors, as well as to subspaces and generalized inverses connected with them.

AMS classification: 15A57; 15A24; 15A27

Keywords: Oblique projector; Orthogonal projector; Idempotent matrix; Hermitian idempotent matrix; Product of projectors; Generalized inverse

* Corresponding author.

E-mail addresses: jbaks@lord.wsp.zgora.pl (J.K. Baksalary), baxx@main.amu.edu.pl (O.M. Baksalary).

PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 3 5 0 - 0

1. Introduction

Two specific subsets of the set $\mathscr{C}_{n,n}$ of $n \times n$ complex matrices are of interest in this paper: the subset $\mathscr{P}$ of all projectors and the subset $\mathscr{P}_{\perp}$ of all orthogonal (with respect to the standard inner product) projectors, which may be characterized as the collections of idempotent matrices and Hermitian idempotent matrices, respectively, i.e.,

$$\mathscr{P} = \{\mathbf{P} \in \mathscr{C}_{n,n} : \mathbf{P} = \mathbf{P}^2\} \quad \text{and} \quad \mathscr{P}_{\perp} = \{\mathbf{P} \in \mathscr{C}_{n,n} : \mathbf{P} = \mathbf{P}^2, \mathbf{P} = \mathbf{P}^*\}, \tag{1.1}$$

where $\mathbf{P}^*$ is the conjugate transpose of $\mathbf{P}$. It is clear that if $\mathbf{Q} = \mathbf{I} - \mathbf{P}$, where $\mathbf{I}$ is the identity matrix of order n, then

$$\mathbf{Q} \in \mathscr{P} \Leftrightarrow \mathbf{P} \in \mathscr{P} \quad \text{and} \quad \mathbf{Q} \in \mathscr{P}_{\perp} \Leftrightarrow \mathbf{P} \in \mathscr{P}_{\perp}. \tag{1.2}$$

Other standard symbols used in this paper are $\mathscr{R}(\mathbf{A})$ and $\mathscr{N}(\mathbf{A})$, which denote the range and null space of an $m \times n$ matrix $\mathbf{A} \in \mathscr{C}_{m,n}$, and $\mathbf{A}\{1\}$, which stands for the set of generalized inverses of $\mathbf{A}$, i.e.,

$$\mathbf{A}\{1\} = \{\mathbf{G} \in \mathscr{C}_{n,m} : \mathbf{AGA} = \mathbf{A}\}. \tag{1.3}$$

A necessary condition for the sum and difference of any projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ to be also projectors is the commutativity property

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1. \tag{1.4}$$

Actually it is known (cf. Theorem in [4, Section 42] and Theorems 5.1.2 and 5.1.3 in [5]) that

$$\mathbf{P}_1 + \mathbf{P}_2 \in \mathscr{P} \Leftrightarrow \mathbf{P}_1\mathbf{P}_2 = \mathbf{0} = \mathbf{P}_2\mathbf{P}_1,$$
$$\mathbf{P}_1 - \mathbf{P}_2 \in \mathscr{P} \Leftrightarrow \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1.$$

The situation changes when considering the products of $\mathbf{P}_1$ and $\mathbf{P}_2$. Then it is known (cf. Theorem in [4, Section 42] and Theorem 5.1.4 in [5]) that without additional assumptions on $\mathbf{P}_1, \ \mathbf{P}_2 \in \mathscr{P}$ we only have

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 \quad \Rightarrow \quad \mathbf{P}_1\mathbf{P}_2 \in \mathscr{P} \text{ and } \mathbf{P}_2\mathbf{P}_1 \in \mathscr{P}. \tag{1.5}$$

Implication (1.5) is in general not reversible, and this well-known fact is confirmed by Example 1 in [3]; see also a collection of examples constructed in this paper in proofs of Theorems 2.1, 3.1, 4.1, and 4.2. However, the commutativity condition in (1.5) becomes both necessary and sufficient when the projectors considered are orthogonal. Since for any $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_{\perp}$ the product $\mathbf{P}_2\mathbf{P}_1$ is the conjugate transpose of $\mathbf{P}_1\mathbf{P}_2$ and $\mathbf{P} \in \mathscr{P} \Leftrightarrow \mathbf{P}^* \in \mathscr{P}$, a modified version of (1.5) can then be expressed in the form

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 \quad \Leftrightarrow \quad \mathbf{P}_1\mathbf{P}_2 \in \mathscr{P} \text{ and } \mathbf{P}_2\mathbf{P}_1 \in \mathscr{P}, \tag{1.6}$$

cf. part (A1) $\Leftrightarrow$ (A7) of Theorem 1 in [1].

The above comparison of statements (1.5) and (1.6) was an inspiration for wider investigations of similarities and dissimilarities between several results on orthogonal projectors and their counterparts corresponding to arbitrary projectors, with special emphasis laid on the commutativity property $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$. The results are presented in three groups. Those in Section 2, related to results of Takane and Yanai [6], refer to matrix characterizations of projectors given in (1.1); those in Section 3, related to results of Groß and Trenkler [3], refer to subspaces connected with projectors according to the fact that if $\mathbf{P} \in \mathscr{P}$, then $\mathbf{P}$ projects onto $\mathscr{R}(\mathbf{P})$ along $\mathscr{N}(\mathbf{P})$; and those in Section 4 refer to specific forms of generalized inverses of the sum $\mathbf{P}_1 + \mathbf{P}_2$ and the difference $\mathbf{P}_1 - \mathbf{P}_2$. The paper contains also a substantial number of numerical examples. To assure completeness of considerations they are devised in every situation, in which a result for orthogonal projectors discontinues to be valid when the orthogonality assumption is violated.

2. Results referring to matrix representations of projectors

Considerations of this paper are concerned with the idempotency of eight nontrivial products of two projectors from among $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{Q}_1 = \mathbf{I} - \mathbf{P}_1$, and $\mathbf{Q}_2 = \mathbf{I} - \mathbf{P}_2$, and the first theorem refers to characterizations of these projectors according to (1.1) and (1.2). A crucial role in the proof of its part concerning orthogonal projectors is played by the following lemma, which will also be employed in proofs of further results.

Lemma. *For any* $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}$, *let* $\mathbf{P}_{ij} = \mathbf{P}_i\mathbf{P}_j$, $\mathbf{P}_{iji} = \mathbf{P}_i\mathbf{P}_j\mathbf{P}_i$, $\mathbf{P}_{ijij} = \mathbf{P}_i\mathbf{P}_j\mathbf{P}_i\mathbf{P}_j$, *and* $\mathbf{P}_{ijiji} = \mathbf{P}_i\mathbf{P}_j\mathbf{P}_i\mathbf{P}_j\mathbf{P}_i$, $i, j = 1, 2;\ i \neq j$. *If* $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$, *then each equality between any two of the products specified above is equivalent to the commutativity property* $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$.

Proof. First observe that if $\mathbf{P}_{12} = \mathbf{P}_{21}$, then each of eight products specified in the lemma coincides with $\mathbf{P}_{12}$, which shows that property (1.4) is a sufficient condition. On the other hand notice that from among 27 possible equalities, which are to be considered in addition to $\mathbf{P}_{12} = \mathbf{P}_{21}$, there are 12 having their counterparts obtained by interchanging $\mathbf{P}_1$ and $\mathbf{P}_2$, and therefore it remains to examine 15 cases. From each of the following four equalities:

$$\mathbf{P}_{12} = \mathbf{P}_{121},\quad \mathbf{P}_{12} = \mathbf{P}_{212},\quad \mathbf{P}_{12} = \mathbf{P}_{12121},\quad \mathbf{P}_{12} = \mathbf{P}_{21212},$$

property (1.4) is obtained directly by taking the conjugate transposes and noting that $\mathbf{P}_{12}^* = \mathbf{P}_{21}$ and the matrices on the right-hand sides are Hermitian. Further, using similar arguments it can easily be verified that

$$(\mathbf{P}_{12} - \mathbf{P}_{121})(\mathbf{P}_{12} - \mathbf{P}_{121})^* = \mathbf{P}_{121} - \mathbf{P}_{12121}. \tag{2.1}$$

On account of the fact that, for any matrix $\mathbf{A} \in \mathscr{C}_{m,n}$,

$$\mathbf{A}\mathbf{A}^* = \mathbf{0} \ \Leftrightarrow\ \mathbf{A} = \mathbf{0}, \tag{2.2}$$

a consequence of (2.1) is

$$\mathbf{P}_{12} = \mathbf{P}_{121} \;\Leftrightarrow\; \mathbf{P}_{121} = \mathbf{P}_{12121}. \tag{2.3}$$

Since the condition on the left-hand side of (2.3) is already known to be equivalent to (1.4) and since the condition on the right-hand side of (2.3) follows straightforwardly from each of the equalities:

$$\mathbf{P}_{12} = \mathbf{P}_{1212},\quad \mathbf{P}_{121} = \mathbf{P}_{212},\quad \mathbf{P}_{121} = \mathbf{P}_{1212},\quad \mathbf{P}_{121} = \mathbf{P}_{2121},$$

next five required implications are established. Finally, it is clear that

$$(\mathbf{P}_{121} - \mathbf{P}_{212})(\mathbf{P}_{121} - \mathbf{P}_{212})^* = \mathbf{P}_{12121} - \mathbf{P}_{121212} - \mathbf{P}_{212121} + \mathbf{P}_{21212}. \tag{2.4}$$

In view of (2.2), a consequence of (2.4) is

$$\mathbf{P}_{121} = \mathbf{P}_{212} \;\Leftrightarrow\; \mathbf{P}_{12121} - \mathbf{P}_{121212} + \mathbf{P}_{21212} - \mathbf{P}_{212121} = \mathbf{0}. \tag{2.5}$$

Since the condition on the left-hand side of (2.5) has already been shown to be equivalent to (1.4) and since the condition on the right-hand side of (2.5) follows from each of the equalities:

$$\mathbf{P}_{12} = \mathbf{P}_{2121},\quad \mathbf{P}_{121} = \mathbf{P}_{21212},\quad \mathbf{P}_{1212} = \mathbf{P}_{2121},$$
$$\mathbf{P}_{1212} = \mathbf{P}_{12121},\quad \mathbf{P}_{1212} = \mathbf{P}_{21212},\quad \mathbf{P}_{12121} = \mathbf{P}_{21212},$$

the last six implications are established, thus concluding the proof. □

Theorem 2.1. *For any* $\mathbf{P}_i \in \mathscr{P}$ *and* $\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_i$, $i = 1, 2$, *consider the following eight statements*:

(S1) $\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P}$, (S1′) $\mathbf{P}_2\mathbf{P}_1 \in \mathscr{P}$,
(S2) $\mathbf{P}_1\mathbf{Q}_2 \in \mathscr{P}$, (S2′) $\mathbf{Q}_2\mathbf{P}_1 \in \mathscr{P}$,
(S3) $\mathbf{Q}_1\mathbf{P}_2 \in \mathscr{P}$, (S3′) $\mathbf{P}_2\mathbf{Q}_1 \in \mathscr{P}$,
(S4) $\mathbf{Q}_1\mathbf{Q}_2 \in \mathscr{P}$, (S4′) $\mathbf{Q}_2\mathbf{Q}_1 \in \mathscr{P}$.

If $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$, *then statements* (S1)–(S4′) *are mutually equivalent and a necessary and sufficient condition for them is that* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *commute. If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are arbitrary projectors*, *then none of these statements is equivalent to any other*, *and the commutativity property*, *while still being sufficient*, *is no longer necessary. However*, *it becomes both necessary and sufficient in* 16 *cases when any pair of statements from among combinations* (S1) *with* (S1′), (S4) *with* (S4′), (S1) *with* (S4), *and* (S1′) *with* (S4′) *holds along with any pair of statements from among combinations* (S2) *with* (S2′), (S3) *with* (S3′), (S2) *with* (S3), *and* (S2′) *with* (S3′).

Proof. In view of the specification of $\mathscr{P}$ in (1.1), statements (S1)–(S4′) can be characterized with the use of the notation introduced in Lemma as follows:

$$\text{(S1)} \Leftrightarrow \mathbf{P}_{12} = \mathbf{P}_{1212}, \qquad \text{(S1}'\text{)} \Leftrightarrow \mathbf{P}_{21} = \mathbf{P}_{2121}, \tag{2.6}$$
$$\text{(S2)} \Leftrightarrow \mathbf{P}_{121} = \mathbf{P}_{1212}, \qquad \text{(S2}'\text{)} \Leftrightarrow \mathbf{P}_{121} = \mathbf{P}_{2121}, \tag{2.7}$$
$$\text{(S3)} \Leftrightarrow \mathbf{P}_{212} = \mathbf{P}_{1212}, \qquad \text{(S3}'\text{)} \Leftrightarrow \mathbf{P}_{212} = \mathbf{P}_{2121}, \tag{2.8}$$

$$\begin{aligned}&(\mathrm{S4}) \;\Leftrightarrow\; \mathbf{P}_{21} - \mathbf{P}_{121} = \mathbf{P}_{212} - \mathbf{P}_{1212},\\ &(\mathrm{S4}') \;\Leftrightarrow\; \mathbf{P}_{12} - \mathbf{P}_{212} = \mathbf{P}_{121} - \mathbf{P}_{2121}.\end{aligned} \tag{2.9}$$

From (2.6)–(2.9) it is seen that $\mathbf{P}_{12} = \mathbf{P}_{21}$ is a sufficient condition for all the statements involved, irrespective of whether $\mathbf{P}_1$ and $\mathbf{P}_2$ are orthogonal or not. If $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$, then – in view of Lemma – an immediate consequence of (2.6)–(2.8) is that each of the statements (S1)–(S3′) is actually equivalent to the commutativity of $\mathbf{P}_1$ and $\mathbf{P}_2$. The equivalence of (S4) to this property follows by noting that postmultiplying the former equality in (2.9) by $\mathbf{P}_1$ yields

$$\mathbf{P}_{21} - \mathbf{P}_{121} = \mathbf{P}_{2121} - \mathbf{P}_{12121},$$

and combining these two equalities shows that

$$(\mathbf{P}_{21} - \mathbf{P}_{121})(\mathbf{P}_{21} - \mathbf{P}_{121})^* = \mathbf{P}_{212} - \mathbf{P}_{2121} - \mathbf{P}_{1212} + \mathbf{P}_{12121} = \mathbf{0}.$$

Hence, on account of (2.2) and Lemma, it follows that (S4) $\Leftrightarrow \mathbf{P}_{21} = \mathbf{P}_{121} \Leftrightarrow \mathbf{P}_{12} = \mathbf{P}_{21}$. The same conclusion is valid for (S4′) due to the fact that if $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$, then $\mathbf{Q}_1\mathbf{Q}_2 \in \mathscr{P}$ if and only if $(\mathbf{Q}_1\mathbf{Q}_2)^* = \mathbf{Q}_2\mathbf{Q}_1 \in \mathscr{P}$.

The second part of the proof consists in providing appropriate examples. The projectors given by Groß and Trenkler [3, p. 254] indicate the possibility of having $\mathbf{P}_{12} \in \mathscr{P}$ and, simultaneously, $\mathbf{P}_{21} \notin \mathscr{P}$. This implies that there is no implication-type relationship between (S1) and (S1′), (S2) and (S2′), (S3) and (S3′), and (S4) and (S4′). The remaining independence conclusions follow by considering projectors

$$\mathbf{P}^{(1)} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \quad \mathbf{P}^{(2)} = \begin{pmatrix} 1 & 1 & 0\\ 0 & 0 & 0\\ 1 & 1 & 0\end{pmatrix}, \quad \mathbf{P}^{(3)} = \begin{pmatrix} 1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.10}$$

and

$$\mathbf{P}^{(4)} = \begin{pmatrix} 1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \mathbf{P}^{(5)} = \begin{pmatrix} 0 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \mathbf{P}^{(6)} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}. \tag{2.11}$$

The matrices $\mathbf{P}_1 = \mathbf{P}^{(1)}$ and $\mathbf{P}_2 = \mathbf{P}^{(4)}$ satisfy (S1) and (S1′), but do not fulfill any of the equalities in (2.7), (2.8), and (2.9); the matrices $\mathbf{P}_1 = \mathbf{P}^{(2)}$ and $\mathbf{P}_2 = \mathbf{P}^{(5)}$ satisfy (S2) and (S2′), but do not fulfill any of the equalities in (2.6), (2.8), and (2.9); the matrices $\mathbf{P}_1 = \mathbf{P}^{(5)}$ and $\mathbf{P}_2 = \mathbf{P}^{(2)}$ satisfy (S3) and (S3′), but do not fulfill any of the equalities in (2.6), (2.7), and (2.9); and the matrices $\mathbf{P}_1 = \mathbf{P}^{(3)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$ satisfy (S4) and (S4′), but do not fulfill any of the equalities in (2.6), (2.7), and (2.8).

In all examples presented above $\mathbf{P}_1$ and $\mathbf{P}_2$ yield $\mathbf{P}_{12} \neq \mathbf{P}_{21}$, which shows that if $\mathbf{P}_1, \mathbf{P}_2$ are arbitrary projectors, then their commutativity is no longer a necessary condition for statements (S1)–(S4′). However, since each of the pairs (S2), (S2′) and (S3), (S3′) implies $\mathbf{P}_{1212} = \mathbf{P}_{2121}$, each of the pairs (S2), (S3) and (S2′), (S3′) implies $\mathbf{P}_{121} = \mathbf{P}_{212}$, and

$$\mathbf{P}_{121} = \mathbf{P}_{212} \;\Rightarrow\; \mathbf{P}_{121} = \mathbf{P}_{1212}, \qquad \mathbf{P}_{121} = \mathbf{P}_{2121} \;\Rightarrow\; \mathbf{P}_{1212} = \mathbf{P}_{2121},$$

it is clear that if the pair (S1), (S1′) holds along with (S2), (S2′) or (S3), (S3′) or (S2), (S3) or (S2′), (S3′), then $\mathbf{P}_{12} = \mathbf{P}_{21}$. The same conclusion is valid in the next four cases, in which the pair (S1), (S1′) is replaced by the pair (S4), (S4′), for an obvious consequence of the latter is that

$$\mathbf{P}_{12} - \mathbf{P}_{1212} = \mathbf{P}_{21} - \mathbf{P}_{2121}.$$

Finally observe that each of the pairs (S1), (S4) and (S1′), (S4′) implies that

$$\mathbf{P}_{12} + \mathbf{P}_{21} = \mathbf{P}_{121} + \mathbf{P}_{212}. \tag{2.12}$$

Since premultiplying (2.12) by $\mathbf{P}_1$ and $\mathbf{P}_2$ shows that

$$\mathbf{P}_{12} + \mathbf{P}_{21} = \mathbf{P}_{121} + \mathbf{P}_{212} \quad \Rightarrow \quad \mathbf{P}_{12} = \mathbf{P}_{1212} \text{ and } \mathbf{P}_{21} = \mathbf{P}_{2121}, \tag{2.13}$$

it follows that the pair (S1), (S4) implies (S1′) and the pair (S1′), (S4′) implies (S1). Consequently, the conclusion that $\mathbf{P}_{12} = \mathbf{P}_{21}$ in the cases when either of the pairs (S1), (S4) and (S1′), (S4′) is combined with (S2), (S2′) or (S3), (S3′) or (S2), (S3) or (S2′), (S3′) is achieved on account of the same conclusion, which has earlier been established in regard to the pair (S1), (S1′). □

The assertions that each of the quadruplets (S1), (S2), (S3), (S4) and (S1′), (S2′), (S3′), (S4′) is equivalent to the commutativity of $\mathbf{P}_1$ and $\mathbf{P}_2$ constitute a part of Theorem 3 of Takane and Yanai [6]. (Parenthetically notice that condition (iii) in the above-mentioned theorem can actually be reduced to the equality $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$, for it clearly implies $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_2\mathbf{P}_1\mathbf{P}_2$ and the fact that $\mathbf{P}_1\mathbf{P}_2$ then projects onto $\mathscr{R}(\mathbf{P}_1) \cap \mathscr{R}(\mathbf{P}_2)$ along $\mathscr{N}(\mathbf{P}_1) + \mathscr{N}(\mathbf{P}_2)$.) Additional quadruplets composed from statements (S1)–(S4′), which are equivalent to the commutativity property, can be developed by applying other results of Takane and Yanai [6]: the parts (i) $\Leftrightarrow$ (vi) of Theorem 1 and Corollaries 4–6 and the parts (i) $\Leftrightarrow$ (iv) of Theorem 2 and Corollaries 7–9. However, they do not exhaust the complete list of quadruplets having the desired property, which has been established in Theorem 2.1.

In Theorem 1 and Corollaries 4–6 Takane and Yanai [6] derived conditions, which are equivalent to various pairs composed of statements (S1)–(S4′). However, their results do not contain such a characterization of the pair (S1), (S1′), i.e., the requirement that simultaneously $\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P}$ and $\mathbf{P}_2\mathbf{P}_1 \in \mathscr{P}$, which is basic from the point of view of considering the commutativity property (1.4). It seems interesting to supplement Takane and Yanai's study of oblique projectors by the following.

Theorem 2.2. *For any* $\mathbf{P}_1$, $\mathbf{P}_2 \in \mathscr{P}$, *both products* $\mathbf{P}_1\mathbf{P}_2$ *and* $\mathbf{P}_2\mathbf{P}_1$ *are projectors if and only if*

$$\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 \in \mathscr{P} \tag{2.14}$$

or, *alternatively*, *if and only if*

$$\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P}. \tag{2.15}$$

Proof. In view of (1.1), it can be verified that

$$\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{121} \in \mathscr{P} \quad \Leftrightarrow \quad \mathbf{P}_{12} - \mathbf{P}_{12}^2 + \mathbf{P}_{21} - \mathbf{P}_{21}^2 = \mathbf{P}_{121} - \mathbf{P}_{12121}, \tag{2.16}$$

$$\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_{212} \in \mathscr{P} \quad \Leftrightarrow \quad \mathbf{P}_{12} - \mathbf{P}_{12}^2 + \mathbf{P}_{21} - \mathbf{P}_{21}^2 = \mathbf{P}_{212} - \mathbf{P}_{21212}, \tag{2.17}$$

while on the other hand

$$\mathbf{P}_{12} \in \mathscr{P} \text{ and } \mathbf{P}_{21} \in \mathscr{P} \quad \Leftrightarrow \quad \mathbf{P}_{12} - \mathbf{P}_{12}^2 = \mathbf{0} \text{ and } \mathbf{P}_{21} - \mathbf{P}_{21}^2 = \mathbf{0}. \tag{2.18}$$

Hence it is clear that the equalities in (2.18) follow from the equality in (2.16) by firstly premultiplying and then postmultiplying it by $\mathbf{P}_1$ as well as from the equality in (2.17) by firstly postmultiplying and then premultiplying it by $\mathbf{P}_2$, thus showing that (2.14) $\Rightarrow$ (S1), (S1′) and (2.15) $\Rightarrow$ (S1), (S1′). Since on the other hand the equalities in (2.18) obviously imply the equalities in (2.16) and (2.17), the above implications may be strengthened to the equivalences

$$(2.14) \;\Leftrightarrow\; (\text{S1}), (\text{S1}') \;\Leftrightarrow\; (2.15),$$

which concludes the proof. □

3. Results referring to subspaces connected with projectors

For orthogonal projectors $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$ it is known that the product $\mathbf{P}_1\mathbf{P}_2$ is a projector if and only if it is the orthogonal projector onto $\mathscr{R}(\mathbf{P}_1) \cap \mathscr{R}(\mathbf{P}_2)$ (and, consequently, along $\mathscr{N}(\mathbf{P}_1) + \mathscr{N}(\mathbf{P}_2)$) and an equivalent condition is the inclusion $\mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$; cf. part (A7) $\Leftrightarrow$ (A11) $\Leftrightarrow$ (A22) of Theorem 1 in [1]. Moreover, it has already been emphasized that another necessary and sufficient condition is the commutativity property $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$. When the orthogonality assumption is deleted, the results change substantially. For arbitrary projectors $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}$ it is known (cf. Theorem 5.1.4 in [5]) that if (1.4) holds, then $\mathbf{P}_1\mathbf{P}_2$ is the projector onto $\mathscr{R}(\mathbf{P}_1) \cap \mathscr{R}(\mathbf{P}_2)$ along $\mathscr{N}(\mathbf{P}_1) + \mathscr{N}(\mathbf{P}_2)$. But it is no longer true that if $\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P}$, then $\mathscr{R}(\mathbf{P}_1\mathbf{P}_2) = \mathscr{R}(\mathbf{P}_1) \cap \mathscr{R}(\mathbf{P}_2)$. This can be seen from the example employing the projectors $\mathbf{P}_1 = \mathbf{P}^{(6)}$ and $\mathbf{P}_2 = \mathbf{P}^{(5)}$ given in (2.11), for which two subspaces in question have different dimensions:

$$\dim \mathscr{R}(\mathbf{P}_1\mathbf{P}_2) = 2 \quad \text{and} \quad \dim\,[\mathscr{R}(\mathbf{P}_1) \cap \mathscr{R}(\mathbf{P}_2)] = 1.$$

A complete solution to the problem of specifying $\mathscr{R}(\mathbf{P}_1\mathbf{P}_2)$ when $\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P}$ is due to Groß and Trenkler [3]. Their Theorem 1 asserts that

$$\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P} \;\Leftrightarrow\; \mathscr{R}(\mathbf{P}_1\mathbf{P}_2) = \mathscr{R}(\mathbf{P}_1) \cap (\mathscr{R}(\mathbf{P}_2) \oplus [\mathscr{N}(\mathbf{P}_1) \cap \mathscr{N}(\mathbf{P}_2)]),$$

thus strengthening the result

$$\mathbf{P}_1\mathbf{P}_2 \in \mathscr{P} \;\Leftrightarrow\; \mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2) \oplus [\mathscr{N}(\mathbf{P}_1) \cap \mathscr{N}(\mathbf{P}_2)]$$

originally given in [2, p. 339] and explicitly proved in [3] as Lemma 1. It should be added that another necessary and sufficient condition for the product $\mathbf{P}_1\mathbf{P}_2$ to be a projector, based on specification of $\mathscr{R}(\mathbf{P}_2)$, was established by Werner [7, Lemma 2]; see also an alternative proof of this result in [6, Note 2].

Coming back to the mainstream of our considerations we will investigate the conditions $\mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$ and $\mathscr{R}(\mathbf{P}_2\mathbf{P}_1) \subseteq \mathscr{R}(\mathbf{P}_1)$ together with their six modifications, which result from replacing $\mathbf{P}_1, \mathbf{P}_2$ by $\mathbf{Q}_1$ and/or $\mathbf{Q}_2$, respectively. We utilize the fact that if $\mathbf{P} \in \mathscr{P}$ and $\mathbf{Q} = \mathbf{I} - \mathbf{P}$, then

$$(\mathscr{R}(\mathbf{Q}) = \mathscr{N}(\mathbf{P}). \tag{3.1}$$

Theorem 3.1. *For any* $\mathbf{P}_i \in \mathscr{P}$ *and* $\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_i$, $i = 1, 2$, *consider the following eight statements*:

(S5)	$\mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$,	(S5′)	$\mathscr{R}(\mathbf{P}_2\mathbf{P}_1) \subseteq \mathscr{R}(\mathbf{P}_1)$,
(S6)	$\mathscr{R}(\mathbf{P}_1\mathbf{Q}_2) \subseteq \mathscr{N}(\mathbf{P}_2)$,	(S6′)	$\mathscr{R}(\mathbf{Q}_2\mathbf{P}_1) \subseteq \mathscr{R}(\mathbf{P}_1)$,
(S7)	$\mathscr{R}(\mathbf{Q}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$,	(S7′)	$\mathscr{R}(\mathbf{P}_2\mathbf{Q}_1) \subseteq \mathscr{N}(\mathbf{P}_1)$,
(S8)	$\mathscr{R}(\mathbf{Q}_1\mathbf{Q}_2) \subseteq \mathscr{N}(\mathbf{P}_2)$,	(S8′)	$\mathscr{R}(\mathbf{Q}_2\mathbf{Q}_1) \subseteq \mathscr{N}(\mathbf{P}_1)$.

If $\mathbf{P}_1, \mathbf{P}_2$ *are orthogonal projectors*, *then all these statements are mutually equivalent and a necessary and sufficient condition for them is that* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *commute. If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are arbitrary projectors*, *then the statements split into four independent groups*:

$$(\mathrm{S5}) \Leftrightarrow (\mathrm{S7}),\quad (\mathrm{S6}') \Leftrightarrow (\mathrm{S5}'),\quad (\mathrm{S7}') \Leftrightarrow (\mathrm{S8}'),\quad (\mathrm{S8}) \Leftrightarrow (\mathrm{S6}), \tag{3.2}$$

and the commutativity property, *while still being sufficient*, *is no longer necessary. However*, *it becomes both necessary and sufficient in eight cases when either of statements* (S5), (S7) *holds along with either of statements* (S6), (S8) *and when either of statements* (S5′), (S6′) *holds along with either of statements* (S7′), (S8′).

Proof. On account of the definition of the null space and the fact that if $\mathbf{P} \in \mathscr{P}$, then for any $\mathbf{A} \in \mathscr{C}_{n,p}$ the inclusion $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{P})$ is equivalent to $\mathbf{PA} = \mathbf{A}$, it can be verified that

$$(\mathrm{S5}) \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{212} \;\Leftrightarrow\; (\mathrm{S7}), \qquad (\mathrm{S6}') \;\Leftrightarrow\; \mathbf{P}_{21} = \mathbf{P}_{121} \;\Leftrightarrow\; (\mathrm{S5}'), \tag{3.3}$$

$$(\mathrm{S7}') \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{121} \;\Leftrightarrow\; (\mathrm{S8}'), \qquad (\mathrm{S8}) \;\Leftrightarrow\; \mathbf{P}_{21} = \mathbf{P}_{212} \;\Leftrightarrow\; (\mathrm{S6}). \tag{3.4}$$

In view of Lemma, the assertions concerning orthogonal projectors follow immediately from the equivalences in (3.3) and (3.4). These equivalences also prove (3.2) and the sufficiency of (1.4).

Consequently, it remains to show that there is no implication-type relationship between four groups of statements in (3.2) and that the list of eight cases, in which pairs of statements from among (S5) – (S8′) entail the commutativity property $\mathbf{P}_{12} = \mathbf{P}_{21}$, is complete. But this can be seen by examining firstly examples, which utilize the projectors

$$\mathbf{P}^{(7)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}^{(8)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{P}^{(9)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and then examples, which employ the projectors given in (2.10) and (2.11). The matrices $\mathbf{P}_1 = \mathbf{P}^{(7)}$ and $\mathbf{P}_2 = \mathbf{P}^{(8)}$ satisfy (S5), but do not fulfill (S6′), (S7′), and (S8); the matrices $\mathbf{P}_1 = \mathbf{P}^{(7)}$ and $\mathbf{P}_2 = \mathbf{P}^{(9)}$ satisfy (S8), but do not fulfill (S5), (S6′), and (S7′); and next two counterexamples can be obtained by interchanging $\mathbf{P}_1$ with $\mathbf{P}_2$. Moreover, the commutativity property is not implied by the pair (S5), (S6′) when $\mathbf{P}_1 = \mathbf{P}^{(2)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$, by the pair (S5), (S7′) when $\mathbf{P}_1 = \mathbf{P}^{(5)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$, by the pair (S6′), (S8) when $\mathbf{P}_1 = \mathbf{P}^{(6)}$ and $\mathbf{P}_2 = \mathbf{P}^{(5)}$, and by the pair (S7′), (S8) when $\mathbf{P}_1 = \mathbf{P}^{(2)}$ and $\mathbf{P}_2 = \mathbf{P}^{(4)}$. □

Theorem 4 of Groß and Trenkler [3] asserts that the commutativity property (1.4) holds if and only if $\mathbf{P}_1\mathbf{P}_2$ is the projector onto $\mathscr{R}(\mathbf{P}_1) \cap \mathscr{R}(\mathbf{P}_2)$ along $\mathscr{N}(\mathbf{P}_1) + \mathscr{N}(\mathbf{P}_2)$ and $\operatorname{rank}(\mathbf{P}_1\mathbf{P}_2) = \operatorname{rank}(\mathbf{P}_2\mathbf{P}_1)$. This in particular implies that $\mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$ and $\mathscr{N}(\mathbf{P}_1) \subseteq \mathscr{N}(\mathbf{P}_1\mathbf{P}_2)$. On account of (3.1), these two inclusions are equivalent to the equalities

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1\mathbf{P}_2 \quad \text{and} \quad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1. \tag{3.5}$$

As can be seen when substituting $\mathbf{P}_1 = \mathbf{P}^{(5)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$, conditions (3.5) are not sufficient for (1.4). However, such a way of reasoning brings a satisfactory result when the condition on the range of one of the products $\mathbf{P}_1\mathbf{P}_2, \mathbf{P}_2\mathbf{P}_1$ is combined with the condition on the null space of the other product. Using a mixture of arguments which refer to both products is very natural when considerations are focused on the commutativity property $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$. From this point of view, the following theorem seems to be an interesting alternative to the above-mentioned result of Groß and Trenkler.

Theorem 3.2. *For any* $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}$, *the commutativity property* $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$ *holds if and only if* $\mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2)$ *and* $\mathscr{N}(\mathbf{P}_2) \subseteq \mathscr{N}(\mathbf{P}_2\mathbf{P}_1)$ *or, alternatively, if and only if* $\mathscr{R}(\mathbf{P}_2\mathbf{P}_1) \subseteq \mathscr{R}(\mathbf{P}_1)$ *and* $\mathscr{N}(\mathbf{P}_1) \subseteq \mathscr{N}(\mathbf{P}_1\mathbf{P}_2)$.

Proof. In view of (3.1), it follows that

$$\begin{aligned} \mathscr{R}(\mathbf{P}_1\mathbf{P}_2) \subseteq \mathscr{R}(\mathbf{P}_2) \;&\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{212}, \\ \mathscr{N}(\mathbf{P}_2) \subseteq \mathscr{N}(\mathbf{P}_{21}) \;&\Leftrightarrow\; \mathbf{P}_{21} = \mathbf{P}_{212}, \end{aligned} \tag{3.6}$$

which implies $\mathbf{P}_{12} = \mathbf{P}_{21}$. Sufficiency of this property for the equalities in (3.6) is obvious. An alternative formulation is obtained by interchanging $\mathbf{P}_1$ with $\mathbf{P}_2$. □

4. Results referring to generalized inverses connected with projectors

A surprising characterization of the commutativity of orthogonal projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ is a specific form of a generalized inverse of their sum $\mathbf{P}_1 + \mathbf{P}_2$; cf. part (A1) $\Leftrightarrow$ (A18) of Theorem 1 in [1]. From the first part of Theorem 2.1 it follows immediately that, under the orthogonality assumption, equivalent characterizations can be obtained by interchanging $\mathbf{P}_1$ and $\mathbf{P}_2$ and replacing $\mathbf{P}_1$ by $\mathbf{Q}_1 = \mathbf{I} - \mathbf{P}_1$ and/or $\mathbf{P}_2$ by $\mathbf{Q}_2 = \mathbf{I} - \mathbf{P}_2$. All these characterizations are comprised in Theorem 4.1, which also reveals changes in the corresponding results when $\mathbf{P}_1$ and $\mathbf{P}_2$ are allowed to be arbitrary projectors.

Theorem 4.1. *For any* $\mathbf{P}_i \in \mathscr{P}$ *and* $\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_i$, $i = 1, 2$, *consider the following eight statements*:

$$
\begin{array}{ll}
(\mathrm{S9}) & \mathbf{P}_1 + \mathbf{P}_2 - \tfrac{3}{2}\mathbf{P}_1\mathbf{P}_2 \in (\mathbf{P}_1 + \mathbf{P}_2)\{1\},\\
(\mathrm{S9}') & \mathbf{P}_1 + \mathbf{P}_2 - \tfrac{3}{2}\mathbf{P}_2\mathbf{P}_1 \in (\mathbf{P}_1 + \mathbf{P}_2)\{1\},\\
(\mathrm{S10}) & \mathbf{P}_1 + \mathbf{Q}_2 - \tfrac{3}{2}\mathbf{P}_1\mathbf{Q}_2 \in (\mathbf{P}_1 + \mathbf{Q}_2)\{1\},\\
(\mathrm{S10}') & \mathbf{P}_1 + \mathbf{Q}_2 - \tfrac{3}{2}\mathbf{Q}_2\mathbf{P}_1 \in (\mathbf{P}_1 + \mathbf{Q}_2)\{1\},\\
(\mathrm{S11}) & \mathbf{Q}_1 + \mathbf{P}_2 - \tfrac{3}{2}\mathbf{Q}_1\mathbf{P}_2 \in (\mathbf{Q}_1 + \mathbf{P}_2)\{1\},\\
(\mathrm{S11}') & \mathbf{Q}_1 + \mathbf{P}_2 - \tfrac{3}{2}\mathbf{P}_2\mathbf{Q}_1 \in (\mathbf{Q}_1 + \mathbf{P}_2)\{1\},\\
(\mathrm{S12}) & \mathbf{Q}_1 + \mathbf{Q}_2 - \tfrac{3}{2}\mathbf{Q}_1\mathbf{Q}_2 \in (\mathbf{Q}_1 + \mathbf{Q}_2)\{1\},\\
(\mathrm{S12}') & \mathbf{Q}_1 + \mathbf{Q}_2 - \tfrac{3}{2}\mathbf{Q}_2\mathbf{Q}_1 \in (\mathbf{Q}_1 + \mathbf{Q}_2)\{1\}.
\end{array}
$$

If $\mathbf{P}_1$, $\mathbf{P}_2$ *are orthogonal projectors*, *then all these statements are mutually equivalent and a necessary and sufficient condition for them is that* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *commute. If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are arbitrary projectors*, *then the statements split into two independent groups*:

$$
\begin{array}{l}
(\mathrm{S9}) \Leftrightarrow (\mathrm{S9}') \Leftrightarrow (\mathrm{S12}) \Leftrightarrow (\mathrm{S12}'),\\
(\mathrm{S10}) \Leftrightarrow (\mathrm{S10}') \Leftrightarrow (\mathrm{S11}) \Leftrightarrow (\mathrm{S11}'),
\end{array}
\tag{4.1}
$$

and the commutativity property, *while still being sufficient*, *is no longer necessary. However*, *it becomes both necessary and sufficient in* 16 *cases when any condition from among* (S9), (S9′), (S12), (S12′) *holds along with any condition from among* (S10), (S10′), (S11), (S11′).

Proof. Straightforward calculations show that the equality in (1.3) is satisfied by the matrices $\mathbf{A} = \mathbf{P}_1 + \mathbf{P}_2$ and $\mathbf{G} = \mathbf{P}_1 + \mathbf{P}_2 - \frac{3}{2}\mathbf{P}_1\mathbf{P}_2$ if and only if

$$
\mathbf{P}_{12} + 4\mathbf{P}_{21} - \mathbf{P}_{121} - \mathbf{P}_{212} - 3\mathbf{P}_{2121} = \mathbf{0} \tag{4.2}
$$

and by the matrices $\mathbf{A} = \mathbf{Q}_1 + \mathbf{Q}_2$ and $\mathbf{G} = \mathbf{Q}_1 + \mathbf{Q}_2 - \frac{3}{2}\mathbf{Q}_1\mathbf{Q}_2$ if and only if

$$4\mathbf{P}_{12} + \mathbf{P}_{21} - 4\mathbf{P}_{121} - 4\mathbf{P}_{212} + 3\mathbf{P}_{2121} = \mathbf{0}. \tag{4.3}$$

Postmultiplying (4.2) and (4.3) by $\mathbf{P}_1$ leads to $\mathbf{P}_{21} = \mathbf{P}_{2121}$ and therefore

$$\text{(S9)} \;\Leftrightarrow\; \text{(S12)} \;\Leftrightarrow\; \mathbf{P}_{21} = \mathbf{P}_{2121} \;\text{ and }\; \mathbf{P}_{12} + \mathbf{P}_{21} = \mathbf{P}_{121} + \mathbf{P}_{212}. \tag{4.4}$$

Further, since postmultiplying the latter equality in (4.4) by $\mathbf{P}_1$ entails the former one and since $\mathbf{P}_1$ and $\mathbf{P}_2$ occur in $\mathbf{P}_{12} + \mathbf{P}_{21} = \mathbf{P}_{121} + \mathbf{P}_{212}$ symmetrically, it follows that

$$\text{(S9)} \;\Leftrightarrow\; \text{(S9}'\text{)} \;\Leftrightarrow\; \text{(S12)} \;\Leftrightarrow\; \text{(S12}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{12} + \mathbf{P}_{21} = \mathbf{P}_{121} + \mathbf{P}_{212}. \tag{4.5}$$

On the other hand, the equality in (1.3) is satisfied by the matrices $\mathbf{A} = \mathbf{P}_1 + \mathbf{Q}_2$ and $\mathbf{G} = \mathbf{P}_1 + \mathbf{Q}_2 - \frac{3}{2}\mathbf{P}_1\mathbf{Q}_2$ if and only if

$$4\mathbf{P}_{121} - \mathbf{P}_{212} - 3\mathbf{P}_{2121} = \mathbf{0}, \tag{4.6}$$

by the matrices $\mathbf{A} = \mathbf{P}_1 + \mathbf{Q}_2$ and $\mathbf{G} = \mathbf{P}_1 + \mathbf{Q}_2 - \frac{3}{2}\mathbf{Q}_2\mathbf{P}_1$ if and only if

$$4\mathbf{P}_{121} - \mathbf{P}_{212} - 3\mathbf{P}_{1212} = \mathbf{0}, \tag{4.7}$$

by the matrices $\mathbf{A} = \mathbf{Q}_1 + \mathbf{P}_2$ and $\mathbf{G} = \mathbf{Q}_1 + \mathbf{P}_2 - \frac{3}{2}\mathbf{Q}_1\mathbf{P}_2$ if and only if

$$\mathbf{P}_{121} - 4\mathbf{P}_{212} + 3\mathbf{P}_{2121} = \mathbf{0}, \tag{4.8}$$

and by the matrices $\mathbf{A} = \mathbf{Q}_1 + \mathbf{P}_2$ and $\mathbf{G} = \mathbf{Q}_1 + \mathbf{P}_2 - \frac{3}{2}\mathbf{P}_2\mathbf{Q}_1$ if and only if

$$\mathbf{P}_{121} - 4\mathbf{P}_{212} + 3\mathbf{P}_{1212} = \mathbf{0}. \tag{4.9}$$

Since postmultiplying (4.6) and (4.8) by $\mathbf{P}_1$ yields $\mathbf{P}_{121} = \mathbf{P}_{2121}$ and postmultiplying (4.7) and (4.9) by $\mathbf{P}_2$ yields $\mathbf{P}_{212} = \mathbf{P}_{1212}$, it follows that

$$\text{(S10)} \;\Leftrightarrow\; \text{(S11)} \;\Leftrightarrow\; \mathbf{P}_{121} = \mathbf{P}_{212} = \mathbf{P}_{2121}, \tag{4.10}$$

$$\text{(S10}'\text{)} \;\Leftrightarrow\; \text{(S11}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{121} = \mathbf{P}_{212} = \mathbf{P}_{1212}. \tag{4.11}$$

But postmultiplying and premultiplying $\mathbf{P}_{121} = \mathbf{P}_{212}$ by $\mathbf{P}_2$ leads to $\mathbf{P}_{212} = \mathbf{P}_{1212}$ and $\mathbf{P}_{212} = \mathbf{P}_{2121}$, respectively, and therefore (4.10) and (4.11) can be condensed to the form

$$\text{(S10)} \;\Leftrightarrow\; \text{(S10}'\text{)} \;\Leftrightarrow\; \text{(S11)} \;\Leftrightarrow\; \text{(S11}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{121} = \mathbf{P}_{212}. \tag{4.12}$$

It is clear that (4.5) and (4.12) lead immediately to (4.1). Moreover, in view of (2.13), applying Lemma to (4.5) and (4.12) shows that if $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$, then (S9)–(S12$'$) are mutually equivalent and hold if and only if $\mathbf{P}_{12} = \mathbf{P}_{21}$.

The second part of the proof consists in examining two examples, which again employ the projectors given in (2.10) and (2.11). The matrices $\mathbf{P}_1 = \mathbf{P}^{(2)}$ and $\mathbf{P}_2 = \mathbf{P}^{(4)}$ satisfy (S9), but do not fulfill the equality in (4.12), and conversely, the matrices $\mathbf{P}_1 = \mathbf{P}^{(5)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$ satisfy (S10), but do not fulfill the equality in (4.5).

In both examples above $\mathbf{P}_1$ and $\mathbf{P}_2$ yield $\mathbf{P}_{12} \neq \mathbf{P}_{21}$, which shows that if $\mathbf{P}_1, \mathbf{P}_2$ are arbitrary projectors, then their commutativity is no longer a necessary condition for statements (S9)–(S13$'$). However, if (4.5) is combined with (4.12), then on account of (2.13) the former takes the form $\mathbf{P}_{12} = \mathbf{P}_{121} = \mathbf{P}_{21}$, thus completing the proof. $\square$

It appears that characterizations of the commutativity property $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1$, similar to those in Theorem 4.1, can also be derived with the use of a specific form of generalized inverses of differences of the projectors discussed in this paper. Such characterizations have not yet been considered even for orthogonal projectors.

Theorem 4.2. *For any* $\mathbf{P}_i \in \mathscr{P}$ *and* $\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_i$, $i = 1, 2$, *consider the following eight statements*:

$$
\begin{array}{ll}
\text{(S13)} & \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \in (\mathbf{P}_1 - \mathbf{P}_2)\{1\}, \\
\text{(S13}'\text{)} & \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1 \in (\mathbf{P}_1 - \mathbf{P}_2)\{1\}, \\
\text{(S14)} & \mathbf{P}_1 - \mathbf{Q}_2 - \mathbf{P}_1\mathbf{Q}_2 \in (\mathbf{P}_1 - \mathbf{Q}_2)\{1\}, \\
\text{(S14}'\text{)} & \mathbf{P}_1 - \mathbf{Q}_2 - \mathbf{Q}_2\mathbf{P}_1 \in (\mathbf{P}_1 - \mathbf{Q}_2)\{1\}, \\
\text{(S15)} & \mathbf{Q}_1 - \mathbf{P}_2 - \mathbf{Q}_1\mathbf{P}_2 \in (\mathbf{Q}_1 - \mathbf{P}_2)\{1\}, \\
\text{(S15}'\text{)} & \mathbf{Q}_1 - \mathbf{P}_2 - \mathbf{P}_2\mathbf{Q}_1 \in (\mathbf{Q}_1 - \mathbf{P}_2)\{1\}, \\
\text{(S16)} & \mathbf{Q}_1 - \mathbf{Q}_2 - \mathbf{Q}_1\mathbf{Q}_2 \in (\mathbf{Q}_1 - \mathbf{Q}_2)\{1\}, \\
\text{(S16}'\text{)} & \mathbf{Q}_1 - \mathbf{Q}_2 - \mathbf{Q}_2\mathbf{Q}_1 \in (\mathbf{Q}_1 - \mathbf{Q}_2)\{1\}.
\end{array}
$$

If $\mathbf{P}_1$, $\mathbf{P}_2$ *are orthogonal projectors*, *then all these statements are mutually equivalent and a necessary and sufficient condition for them is that* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *commute. If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are arbitrary projectors*, *then the statements split into four independent groups*

$$
\begin{array}{ll}
\text{(S13)} \Leftrightarrow \text{(S16}'\text{)}, & \text{(S14)} \Leftrightarrow \text{(S15}'\text{)}, \\
\text{(S15)} \Leftrightarrow \text{(S14}'\text{)}, & \text{(S16)} \Leftrightarrow \text{(S13}'\text{)},
\end{array}
\tag{4.13}
$$

and the commutativity property, *while still being sufficient*, *is no longer necessary. However*, *it becomes both necessary and sufficient in* 24 *cases when any statement from one of the pairs in* (4.13) *holds along with any statement from another pair.*

Proof. Straightforward calculations show that the equality in (1.3) is satisfied by the matrices $\mathbf{A} = \mathbf{P}_1 - \mathbf{P}_2$ and $\mathbf{G} = \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2$ if and only if

$$\mathbf{P}_{12} - 2\mathbf{P}_{121} + \mathbf{P}_{2121} = \mathbf{0} \tag{4.14}$$

and by the matrices $\mathbf{A} = \mathbf{Q}_1 - \mathbf{Q}_2$ and $\mathbf{G} = \mathbf{Q}_1 - \mathbf{Q}_2 - \mathbf{Q}_2\mathbf{Q}_1$ if and only if

$$\mathbf{P}_{12} - \mathbf{P}_{121} + \mathbf{P}_{212} - \mathbf{P}_{1212} = \mathbf{0}. \tag{4.15}$$

Postmultiplying (4.14) and premultiplying (4.15) by $\mathbf{P}_1$ leads to $\mathbf{P}_{121} = \mathbf{P}_{2121}$ and $\mathbf{P}_{12} = \mathbf{P}_{121}$, respectively, and therefore

$$
\begin{array}{l}
\text{(S13)} \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{121} = \mathbf{P}_{2121}, \\
\text{(S16}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{121},\; \mathbf{P}_{212} = \mathbf{P}_{1212},
\end{array}
$$

which can further be simplified to the form

$$\text{(S13)} \;\Leftrightarrow\; \text{(S16}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{121} = \mathbf{P}_{212}. \tag{4.16}$$

The proofs of the equivalences

$$\text{(S14)} \;\Leftrightarrow\; \text{(S15}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{121} \text{ and } \mathbf{P}_{21} = \mathbf{P}_{212}, \tag{4.17}$$

$$\text{(S15)} \;\Leftrightarrow\; \text{(S14}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{12} = \mathbf{P}_{212} \text{ and } \mathbf{P}_{21} = \mathbf{P}_{121}, \tag{4.18}$$

$$\text{(S16)} \;\Leftrightarrow\; \text{(S13}'\text{)} \;\Leftrightarrow\; \mathbf{P}_{21} = \mathbf{P}_{121} = \mathbf{P}_{212} \tag{4.19}$$

follow by similar arguments. It is clear that (4.16)–(4.19) lead immediately to (4.13). Moreover, if $\mathbf{P}_1, \mathbf{P}_2 \in \mathscr{P}_\perp$, then applying Lemma to these four characterizations of statements (S13)–(S16′) shows that they are mutually equivalent and hold if and only if $\mathbf{P}_{12} = \mathbf{P}_{21}$.

To show that in the general case there is no implication-type relationship between statements (S13)–(S16), which represent four groups in (4.13), we consider examples employing once more the projectors given in (2.10) and (2.11). The matrices $\mathbf{P}_1 = \mathbf{P}^{(5)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$ satisfy (S13), but do not fulfill the second equalities in (4.17) and (4.18) and the first equality in (4.19); the matrices $\mathbf{P}_1 = \mathbf{P}^{(2)}$ and $\mathbf{P}_2 = \mathbf{P}^{(4)}$ satisfy (S14), but do not fulfill the second equality in (4.16) and either of the equalities in (4.18) and (4.19); the matrices $\mathbf{P}_1 = \mathbf{P}^{(2)}$ and $\mathbf{P}_2 = \mathbf{P}^{(6)}$ satisfy (S15), but do not fulfill either of the equalities in (4.16) and (4.17) and the second equality in (4.19); and the matrices $\mathbf{P}_1 = \mathbf{P}^{(6)}$ and $\mathbf{P}_2 = \mathbf{P}^{(5)}$ satisfy (S16), but do not fulfill the first equalities in (4.16)–(4.18).

In all examples presented above $\mathbf{P}_1$ and $\mathbf{P}_2$ yield $\mathbf{P}_{12} \neq \mathbf{P}_{21}$, which shows that if $\mathbf{P}_1$, $\mathbf{P}_2$ are arbitrary projectors, then their commutativity is no longer a necessary condition for statements (S13)–(S16′). However, from (4.16)–(4.19) it follows that combining any two from among these statements, except only for the pairs of equivalent statements listed in (4.13), leads directly to the equality $\mathbf{P}_{12} = \mathbf{P}_{21}$, thus completing the proof. □

It seems interesting that although the commutativity of arbitrary projectors $\mathbf{P}_1$ and $\mathbf{P}_2$ is merely sufficient for $\mathbf{P}_1 + \mathbf{P}_2 - \frac{3}{2}\mathbf{P}_1\mathbf{P}_2$ to be a generalized inverse of $\mathbf{P}_1 + \mathbf{P}_2$ and for $\mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2$ to be a generalized inverse of $\mathbf{P}_1 - \mathbf{P}_2$, it becomes necessary and sufficient when these two conditions are required to hold simultaneously. In view of (4.5) and (4.16), this assertion follows immediately by combining (S9) with (S13) and may be expressed in the form

$$\begin{aligned}\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 \;\Leftrightarrow\; & \mathbf{P}_1 + \mathbf{P}_2 - \tfrac{3}{2}\mathbf{P}_1\mathbf{P}_2 \in (\mathbf{P}_1 + \mathbf{P}_2)\{1\} \\ & \text{and } \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \in (\mathbf{P}_1 - \mathbf{P}_2)\{1\}.\end{aligned}$$

Other combinations of the statements specified in Theorems 4.1 and 4.2 which are equivalent to the commutativity property are: (S9) with (S16), (S10) with (S14), (S10) with (S15), and further 28 alternative versions being consequences of replacing on the one hand (S9) and (S10) according to (4.1) and, on the other hand, each of conditions (S13)–(S16) according to (4.13).

Acknowledgements

The authors are very grateful to George P.H. Styan for stimulating and helpful discussions on projectors during his visit to Poznań. The research of both authors was supported in part by the Polish Committee of Scientific Research Grant No. 2 P03A 010 18.

References

[1] J.K. Baksalary, Algebraic characterizations and statistical implications of the commutativity of orthogonal projectors, in: T. Pukkila, S. Puntanen (Eds.), Proceedings of the Second International Tampere Conference in Statistics, University of Tampere, Tampere, Finland, 1987, pp. 113–142.
[2] A.L. Brown, A. Page, Elements of Functional Analysis, Van Nostrand, New York, 1970.
[3] J. Groß, G. Trenkler, On the product of oblique projectors, Linear Multilinear Algebra 44 (1998) 247–259.
[4] P.R. Halmos, Finite-Dimensional Vector Spaces, Van Nostrand, Princeton, NJ, 1958.
[5] C.R. Rao, S.K. Mitra, Generalized Inverse of Matrices and Its Applications, Wiley, New York, 1971.
[6] Y. Takane, H. Yanai, On oblique projectors, Linear Algebra Appl. 289 (1999) 297–310.
[7] H.J. Werner, G-inverses of matrix products, in: S. Schach, G. Trenkler (Eds.), Data Analysis and Statistical Inference, Eul, Bergisch-Gladbach, 1992, pp. 531–546.